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# Graham's pebbling conjecture on product of thorn graphs of complete graphs

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ABSTRACT

The pebbling number of a graph $G, f(G)$, is the least $n$ such that, no matter how $n$ pebbles are placed on the vertices of $G$, we can move a pebble to any vertex by a sequence of pebbling moves, each move taking two pebbles off one vertex and placing one on an adjacent vertex. Let $p_1, p_2, \ldots, p_n$ be positive integers and $G$ be such a graph, $V(G) = n$. The thorn graph of the graph $G$, with parameters $p_1, p_2, \ldots, p_n$, is obtained by attaching $p_i$ new vertices of degree 1 to the vertex $u_i$ of the graph $G$, $i = 1, 2, \ldots, n$. Graham conjectured that for any connected graphs $G$ and $H, f(G \times H) \leq f(G)f(H)$. We show that Graham's conjecture holds true for a thorn graph of the complete graph with every $p_i > 1$ $(i = 1, 2, \ldots, n)$ by a graph with the two-pebbling property. As a corollary, Graham's conjecture holds when $G$ and $H$ are the thorn graphs of the complete graphs with every $p_i > 1$ $(i = 1, 2, \ldots, n)$.



## 1. Introduction

Pebbling in graphs was first studied by Chung [1]. Consider a connected graph with a fixed number of pebbles which are nonnegative integer weights distributed on the vertices. A pebbling move consists of taking two pebbles off one vertex and placing one pebble on an adjacent vertex. Chung defined the pebbling number of a vertex $v$ in a graph $G$ as the smallest number $f(G, v)$ such that from every placement of $f(G, v)$ pebbles, it is possible to move a pebble to $v$ by a sequence of pebbling moves. Then the pebbling number of a graph $G$, denoted by $f(G)$, is the maximum $f(G, v)$ over all the vertices $v$ in $G$. The $t$-pebbling number of a vertex $v$ in a graph $G$ is the smallest number $f_t(G, v)$ with the property that from every placement of $f_t(G, v)$ pebbles on $G$, it is possible to move $t$ pebbles to $v$ by a sequence of pebbling moves.

There are some known results regarding $f(G)$ (see Refs. [1–7]). If one pebble is placed on each vertex other than the vertex $v$, then no pebble can be moved to $v$. Also, if $\omega$ is at distance $d$ from $v$, and $2^d - 1$ pebbles are placed on $\omega$, then no pebble can be moved to $v$. So it is clear that $f(G) \geq \max(|V(G)|, 2^D)$ [1], where $|V(G)|$ is the number of vertices of the graph $G$ and $D$ is the diameter of the graph $G$. Furthermore, we know from [1] that $f(K_n) = n$, where $K_n$ is the complete graph on $n$ vertices, and $f(P_n) = 2^{n-1}$, where $P_n$ is the path on $n$ vertices. Given a configuration of pebbles placed on $G$, let $q$ be the number of vertices with at least one pebble, and let $r$ be the number of vertices with an odd number of pebbles. We say that $G$ satisfies the two-pebbling property (respectively, weak or odd two-pebbling property), if it is possible to move two pebbles to any specified target vertex when the total starting number of pebbles is $2f(G) - q + 1$ (respectively, $2f(G) - r + 1$). Note that any graph which satisfies the two-pebbling property also satisfies the weak or odd two-pebbling property.

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This paper explores the pebbling number of the Cartesian product of the thorn graph of the complete graph with every $p_i > 1$ $(i = 1, 2, \ldots, n)$. The idea for a Cartesian product comes from a conjecture of Graham [1]. This conjecture states that for any graphs $G$ and $H$, $f(G \times H) \leq f(G)f(H)$. There are a few results that verify Graham's conjecture, among them, the conjecture holds for a tree by a tree [2], a cycle by a cycle [3], and a complete graph by a graph with the two-pebbling property [1] and a complete bipartite graph by a graph with the two-pebbling property [4], a fan graph by a fan graph and a wheel graph by a wheel graph [5]. In this paper, we show that Graham's conjecture holds for a thorn graph of the complete graph with every $p_i > 1$ $(i = 1, 2, \ldots, n)$ by a graph with the two-pebbling property.

**Definition 1.1** (*[8]*). Let $p_1, p_2, \ldots, p_n$ be positive integers and $G$ be such a graph, $V(G) = n$. The thorn of the graph $G$, with parameters $p_1, p_2, \ldots, p_n$, is obtained by attaching $p_i$ new vertices of degree 1 to the vertex $u_i$ of the graph $G$ $(i = 1, 2, \ldots, n)$.

The thorn graph of the graph $G$ will be denoted by $G^*$ or by $G^*$ $(p_1, p_2, \ldots, p_n)$, if the respective parameters need to be specified. In this paper, we will consider the thorn graph with every $p_i > 1$ $(i = 1, 2, \ldots, n)$.

**Definition 1.2** (*[9]*). Given a configuration of pebbles placed on $G$, a transmitting subgraph of $G$ is a path $x_1, x_2, \ldots, x_n$ such that there are at least two pebbles on $x_1$ and at least one pebble on each of the other vertices in the path, possibly except $x_n$. In this case, we can transmit a pebble from $x_1$ to $x_n$.

Throughout this paper $G$ will denote a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex $v$ of a graph $G$, $p(v)$ refers to the number of pebbles on $v$.

## 2. Pebbling of $K_n^*$

**Definition 2.1** (*[7]*). Let $T$ be a tree with a specified vertex $v$. $T$ can be viewed as a directed tree denoted by $\overrightarrow{T_v}$ with edges directed toward a specified vertex, also called the root. A path-partition $P = \{\overrightarrow{P_1}, \ldots, \overrightarrow{P_r}\}$ is a set of nonoverlapping directed paths, the union of which is $\overrightarrow{T_v}$. Throughout this paper, unless stated otherwise, we will always assume that $|E(\overrightarrow{P_i})| > |E(\overrightarrow{P_j})|$ whenever $i \leq j$. A path-partition $P = \{\overrightarrow{P_1}, \ldots, \overrightarrow{P_r}\}$ is said to majorize another (say $Q = \{\overrightarrow{P_1'}, \ldots, \overrightarrow{P_r'}\}$) if the non-increasing sequence of its path size majorizes that of the other. That is, if $a_i = |E(\overrightarrow{P_i})|$ and $b_j = |E(\overrightarrow{P_j'})|$, then $(a_1, \ldots, a_r) > (b_1, \ldots, b_t)$ if and only if $a_i > b_i$ where $i = min\{j : a_j \neq b_j\}$. A path-partition of a tree $T$ is said to be maximum if it majorizes all other path-partitions.

**Theorem 2.2** (*[1]*). *The pebbling number $f_k(t, v)$ for a vertex $v$ in a tree $T$ is $k2^{a_1} + 2^{a_2} + \cdots + 2^{a_t} - t + 1$ where $a_1, a_2, \ldots, a_t$ is the sequence of the path (i.e., the number of edges in the path) in a maximum path-partition of $\overrightarrow{T_v}$.*

**Lemma 2.3.** *Suppose $M_n$ is a graph which satisfies the following properties: (1) the subgraph which consists of $v_1, \ldots, v_n, v_{n+1}$ is a $K_{n+1}$, (2) $v_r$ is adjacent to $u_{rj}$ $(r \neq j; j = 1, \ldots, p_r)$. If the number of pebbles on $M_n$ except $v_i$ is at least $2n+4t-3+\Sigma p_j - p_i$, then $t$ pebbles can be moved to $v_i$.*

**Proof.** Give the following distribution of $2n + 4t - 4 + \Sigma p_j - p_i$ pebbles on $M_n$: $p(u_{11}) = 4t - 1$, $p(u_{j1}) = 3$ $(j = 2, \ldots, i-1, i+1, \ldots, n+1)$, $p(u_{rj}) = 1$ $(r = 1, \ldots, i-1, i+1, \ldots, n+1; j = 2, \ldots, p_r)$, then $t$ pebbles can not be moved to $v_i$. Thus if we can move $t$ pebbles to $v_i$, then $f_t$ $(M_n, v_i) > 2n+4t-4+\Sigma p_j - p_i$. If we remove all edges between $v_{j_1}$ $(j_1 \neq i)$ and $v_{j_2}$ $(j_2 \neq i)$, then the remaining graph is a tree $T$. By Theorem 2.2, we know that $f_t(T, v_i) = 2n + 4t - 3 + \Sigma p_j - p_i$. Since $(T, v_i)$ is a spanning subgraph of $(M_n, v_i)$, $f_t(M_n, v_i) \leq f_t(T, v_i)$. Then $f_t(M_n, v_i) \leq 2n + 4t - 3 + \Sigma p_j - p_i$. Hence $f_t(M_n, v_i) = 2n + 4t - 3 + \Sigma p_j - p_i$. □

**Theorem 2.4.** *Let $K_n^*$ be the thorn graph of $K_n$ with $n \geq 2$ vertices. Then*

$$f(K_n^*) = 2(n + 1) + \Sigma p_j.$$

**Proof.** Label the vertices of $K_n$ by $v_1, \ldots, v_n$. Let the vertex $v_i$ of the graph $K_n$ attach to $u_{ij}$ $(j = 1, \ldots, p_i)$. The graph which is composed of these vertices is $K_n^*$. Consider the following distribution of $2n + 1 + \Sigma p_j$ pebbles on $K_n^*$: $p(u_{11}) = 7$, $p(u_{1j}) = 1$ $(j = 2, \ldots, p_1)$, $p(u_{i1}) = 3$ $(i = 2, \ldots, n-1)$, $p(u_{ij}) = 1$ $(i = 2, \ldots, n-1, j = 2, \ldots, p_i)$, $p(u_{nj}) = 1$ $(j = 2, \ldots, p_n)$. Then no pebble can be moved to $u_{n1}$. So $f(K_n^*) > 2n + 1 + \Sigma p_j$. Now let us consider any distribution of $2(n + 1) + \Sigma p_j$ pebbles on $K_n^*$. There are only two types of possible target vertices.

Case 1. Suppose that the target vertex is $v_i$, where $i = 1, 2, \ldots, n$. If $p(u_{ij}) \geq 2$ for some $j$, then we can move one pebble from $u_{ij}$ to $v_i$. We may assume that $p(u_{ij}) < 2$ for all $j$. When these vertices $u_{i1}, \ldots, u_{ip_i}$ and their edges are removed, the remaining graph is $M_{n-1}$. The number of pebbles on $M_{n-1}$ is at least $2(n + 1) + \Sigma p_j - p_i$. Since $2(n + 1) + \Sigma p_j - p_i > 2(n - 1) + 4 \times 1 - 3 + \Sigma p_j - p_i$, by Lemma 2.3, one pebble can be moved to $v_i$.

Case 2. Suppose that the target vertex is $u_{ij}$, where $i = 1, \ldots, n$ and $j = 1, \ldots, p_i$. If $p(v_i) \geq 2$, then we can move one pebble from $v_i$ to $u_{ij}$. Assuming that $p(v_i) < 2$, we may consider the following two subcases.

(2.1) If $p(v_i) = 1$, then we consider the following two sub-subcases.

(2.1.1) If there exists at least one vertex $u_{ij_1}$ $(j_1 \neq j)$ with $p(u_{ij_1}) \geq 2$, then $\{u_{ij_1}, v_i, u_{ij}\}$ forms a transmitting subgraph.

(2.1.2) If $p(u_{ir}) < 2$ for all $r$ $(r \neq j)$, as in the proof of case 1, by Lemma 2.3, one pebble can be moved to $v_i$. So we can move one pebble from $v_i$ to $u_{ij}$.

(2.2) If $p(v_i) = 0$, and if there exist at least two vertices $u_{ij_1}$ $(j_1 \neq j)$, $u_{ij_2}$ $(j_2 \neq j)$ with $p(u_{ij_1}) \geq 2$, $p(u_{ij_2}) \geq 2$ among these vertices $u_{i1}, \ldots, u_{ip_i}$, then we move one pebble from $u_{ij_1}$ to $v_i$. So $\{u_{ij_2}, v_i, u_{ij}\}$ forms a transmitting subgraph. Otherwise, we consider the following three sub-subcases.

(2.2.1) If $p(u_{ij_1}) \geq 4$ for only $j_1$ $(j_1 \neq j)$ and $p(u_{ir}) < 2$ for all $r$ $(r \neq j_1, j)$, then $\{u_{ij_1}, v_i, u_{ij}\}$ forms a transmitting subgraph.

(2.2.2) If $2 \leq p(u_{ij_1}) < 4$ for only $j_1$ $(j_1 \neq j)$ and $p(u_{ir}) < 2$ for all $r$ $(r \neq j_1, j)$, then we can move one pebble from $u_{ij_1}$ to $v_i$. as in the proof of case 1, by Lemma 2.3, one pebble can be moved to $v_i$. So $\{v_i, u_{ij}\}$ forms a transmitting subgraph.

(2.2.3) If $p(u_{ir}) < 2$ for all $r$ $(r \neq j)$, as in the proof of case 1, by Lemma 2.3, two pebbles can be moved to $v_i$. So $\{v_i, u_{ij}\}$ forms a transmitting subgraph. Hence $f(K_n^*) = 2(n+1) + \Sigma p_j$. $\square$

**Theorem 2.5.** *Let $K_n^*$ be the thorn graph of the complete graph $K_n$. Then $K_n^*$ satisfies the two-pebbling property.*

**Proof.** Let $p$ be the number of pebbles on the thorn graph $K_n^*$, $q$ be the number of the vertices with at least one pebble and $p + q = 2[2(n+1) + \Sigma p_j] + 1$. Clearly, $K_n^*$ is a tree when $n = 1$ or $n = 2$. From Ref. [1], we know that a tree satisfies the two-pebbling property. We may assume that $n \geq 3$. Then we consider the following two types of possible target vertices.

Case 1. Suppose that the target vertex is $v_i$, where $i = 1, 2, \ldots, n$. Without loss of generality, we assume that the target vertex is $v_1$. If $p(v_1) = 1$, then the number of pebbles on all the vertices except $v_1$ is $2[2(n+1) + \Sigma p_j] + 1 - q - 1 > 2(n+1) + \Sigma p_j$ (since $q \leq n + \Sigma p_j$). Since $f(K_n^*) = 2(n+1) + \Sigma p_j$, we can put one more pebble on $v_1$ using $2[2(n+1) + \Sigma p_j] + 1 - q - 1$ pebbles. If $p(v_1) = 0$, then we consider the following two subcases.

(1.1) Suppose that $p(u_{1j}) \geq 2$ for some $u_{1j}$. Then we can move one pebble from $u_{1j}$ to $v_1$. Using the remaining $2[2(n+1) + \Sigma p_j] + 1 - q - 2$ pebbles, we can move another pebble to $v_1$.

(1.2) Suppose that $p(u_{1j}) < 2$ for all $u_{1j}$. As in the proof of case 1 of Theorem 2.4, by Lemma 2.3, we can move two pebbles to $v_1$.

Case 2. Suppose that the target vertex is $u_{ij}$, where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, p_i$. Without loss of generality, we assume the target vertex is $u_{11}$. If $p(u_{11}) = 1$, then the number of pebbles on all the vertices except $u_{11}$ is $2[2(n+1) + \Sigma p_j] + 1 - q - 1 > 2(n+1) + \Sigma p_j$ (since $q \leq n + \Sigma p_j$). Since $f(K_n^*) = 2(n+1) + \Sigma p_j$, we can put one more pebble on $u_{11}$ using $2[2(n+1) + \Sigma p_j] + 1 - q - 1$ pebbles. If $p(u_{11}) = 0$, then we consider the following three subcases.

(2.1) If $p(v_1) \geq 2$, then we can move one pebble from $v_1$ to $u_{11}$. Using the remaining $2[2(n+1) + \Sigma p_j] + 1 - q - 2$ pebbles, we can move another pebble to $u_{11}$.

(2.2) If $p(v_1) = 1$, and if there is at least one vertex $u_{1j_1}$ $(j_1 \neq 1)$ with $p(u_{1j_1}) \geq 2$, then $\{u_{1j_1}, v_1, u_{11}\}$ forms a transmitting subgraph. Using the $2[2(n+1)+\Sigma p_j]+1-q-3$ pebbles, we can move another pebble to $u_{11}$. If $p(u_{1r}) < 2$ for all $r$ $(r \neq j)$, as in the proof of case 1 of Theorem 2.4, by Lemma 2.3, we move another three pebbles to $v_1$. So we move two pebbles from $v_1$ to $u_{11}$.

(2.3) If $p(v_1) = 0$, and if there are at least two vertices $u_{1j_1}, u_{1j_2}$ $(j_1, j_2 \neq 1)$ with $p(u_{1j_1}) \geq 2$, $p(u_{1j_2}) \geq 2$, then we can move one pebble from $u_{1j_2}$ to $v_1$. Then $\{u_{1j_1}, v_1, u_{11}\}$ forms a transmitting subgraph. Using the remaining $2[2(n+1)+\Sigma p_j]+1-q-4$ pebbles, we can move another pebble to $u_{11}$. If there is only one vertex $u_{1j_1}$ $(j_1 \neq 1)$ with $p(u_{1j_1}) \geq 4$ and $p(u_{1j}) < 2$ for all $j$ $(j \neq 1, j_1)$, then we can move two pebbles from $u_{1j_1}$ to $v_1$. So $\{v_1, u_{11}\}$ forms a transmitting subgraph. Using the remaining $2[2(n+1) + \Sigma p_j] + 1 - q - 4$ pebbles, we can move another pebble to $u_{11}$. If there is only one vertex $u_{1j_1}$ $(j_1 \neq 1)$ with $3 \geq p(u_{1j_1}) \geq 2$ and for all $j$ $(j \neq 1, j_1)$, then we can move one pebble from $u_{1j_1}$ to $v_1$. And if we delete these vertices $u_{11}, u_{12}, \ldots, u_{1p_1}$, then the remaining graph is $M_{n-1}$. The number of pebbles on $M_{n-1}$ except $v_1$ is at least $2[2(n+1) + \Sigma p_j] + 1 - q - (p_1 + 1)$. Since $q \leq n + \Sigma p_j - 2$, $f(K_n^*) = 2(n+1) + \Sigma p_j$, then $2[2(n+1) + (\Sigma p_j)] + 1 - q - (p_1 + 1) \geq 3n + 6 + \Sigma p_j - p_1 > 2(n-1) + 4 \times 3 + \Sigma p_j - p_1$. By Lemma 2.3, we move another three pebbles to $v_1$. So we move two pebbles from $v_1$ to $u_{11}$. We may assume that $p(u_{1j_1}) < 2$ for all $j$ $(j \neq 1)$. As in the proof of case 1 of Theorem 2.4, by Lemma 2.3, we move four pebbles to $v_1$. So we move two pebbles from $v_1$ to $u_{11}$. $\square$

## 3. Cartesian product

Let $G$ and $H$ be two graphs, the (Cartesian) product of $G$ and $H$, denoted by $G \times H$, is the graph whose vertex set is the Cartesian product

$$V(G \times H) = V(G) \times V(H) = \{(x, y) : x \in V(G), y \in V(H)\}$$

and two vertices $(x, y)$ and $(x', y')$ are adjacent if and only if $x = x'$ and $\{y, y'\} \in E(H)$, or $\{x, x'\} \in E(G)$ and $y = y'$. We can depict $G \times H$ pictorially by drawing a copy of $H$ at every vertex of $G$ and connecting each vertex in one copy of $H$ to the corresponding vertex in an adjacent copy of $H$. We write $\{x\} \times H$ (respectively, $G \times \{y\}$) for the subgraph of vertices whose projection onto $V(G)$ is the vertex $x$ (respectively, whose projection onto $V(H)$ is $y$). If the vertices of $G$ are labeled by $x_i$, then for any distribution of pebbles on $G \times H$, we write $p_i$ for the number of pebbles on $\{x_i\} \times H$, $q_i$ for the number of occupied vertices of $\{x_i\} \times H$ and $r_i$ for the number of vertices of $\{x_i\} \times H$ with an odd number of pebbles.

The following conjecture, by Ronald Graham, suggests a constraint on the pebbling number of the product of two graphs.

**Conjecture** (*Graham*). *The pebbling number of* $G \times H$ *satisfies*

$$f(G \times H) \le f(G)f(H).$$

**Lemma 3.1** ([3]). *Let* $\{x_i, x_j\}$ *be an edge in G. Suppose that in* $G \times H$, *we have* $p_i$ *pebbles on* $\{x_i\} \times H$, *and* $r_i$ *of these vertices have an odd number of pebbles. If* $r_i \le k \le p_i$, *and if k and* $p_i$ *have the same parity, then k pebbles can be retained on* $\{x_i\} \times H$, *while transferring* $\frac{p_i - k}{2}$ *pebbles on to* $\{x_j\} \times H$. *If k and* $p_i$ *have opposite parity, we must leave* $k+1$ *pebbles on* $\{x_i\} \times H$, *so we can only transfer* $\frac{p_i-(k+1)}{2}$ *pebbles onto* $\{x_j\} \times H$. *In particular, we can always transfer* $\frac{p_i - r_i}{2}$ *pebbles on to* $\{x_j\} \times H$, *since* $p_i$ *and* $r_i$ *have the same parity. In all these cases, the number of vertices of* $\{x_i\} \times H$ *with an odd number of pebbles is unchanged by these transfers.*

**Lemma 3.2** ([2]). *Let* $q_1, q_2, \ldots, q_n$ *be the non-increasing sequence of path lengths of a maximum path partition* $Q = \{Q_1, \ldots, Q_m\}$ *of a tree T. Then*

$$f(T) = \left(\sum_{i=1}^{m} 2^{q_i}\right) - m + 1.$$

**Lemma 3.3** ([2]). *If T is a tree, and G satisfies the odd two-pebbling property, then* $f((T, G), (x, y)) \le f(T, x)f(G)$ *for every vertex* $v$ *in G.*

## 4. Pebbling $K_n^* \times K_m^*$

In this section, we show that Graham's conjecture holds for the product of the thorn graph of the complete graph and a graph with the two-pebbling property.

**Theorem 4.1.** *If G satisfies the two-pebbling property, then*

$$f(K_n^* \times G) \le f(K_n^*)f(G).$$

**Proof.** Label the vertices of $K_n$ by $v_1, \ldots, v_n$, and let the new vertex that attaches to the vertex $v_i$ of the graph be $u_{ij}$ ($i = 1, 2, \ldots, n, j = 1, \ldots, p_i$). The graph which is composed of these vertices is $K_n^*$. Let $G_{ij}$ denote the subgraph $\{u_{ij}\} \times G \subsetneq K_n^* \times G$, and $H_i$ denote the subgraph $\{v_i\} \times G \subsetneq K_n^* \times G$. Let $m_{ij}$ denote the number of pebbles on the vertices of $G_{ij}$, and $n_i$ denote the number of pebbles on the vertices of $H_i$. Let $r_{ij}$ denote the number of vertices in $G_{ij}$ which have an odd number of pebbles, and $t_i$ denote the number of vertices in $H_i$ which have an odd number of pebbles. Take any arrangement of $[2(n+1) + \Sigma p_j]f(G)$ pebbles on the vertices of $K_n^* \times G$.

First we assume that the target vertex is $(v_i, y)$ for some $y$, where $i = 1, 2, \ldots, n$. Without loss of generality, we may assume that the vertex is $(v_1, y)$. Let $K_n^* - \{u_{11}, \ldots, u_{1p_1}, u_{21}, \ldots, u_{2p_2}, \ldots, u_{n1}, u_{n2}, \ldots, u_{np_n}\} = K_n$. From ref [1], we know that $f(K_n \times G, (v_1, y)) = f(K_n \times G) \le nf(G)$. Since $r_{ij} \le |V(G)| \le f(G)$, $\sum_{i=1}^{n} \sum_{j=1}^{p_i} m_{ij} \le [2(n+1) + \Sigma p_j]f(G)$, then

$$\begin{aligned}
\sum_{i=1}^{n} \sum_{j=1}^{p_i} (m_{ij} + r_{ij}) &= \sum_{i=1}^{n} \sum_{j=1}^{p_i} m_{ij} + \sum_{i=1}^{n} \sum_{j=1}^{p_i} r_{ij} \\
&\le [2(n+1) + \Sigma p_j]f(G) + \Sigma p_j f(G) \\
&= [2(n+1) + 2\Sigma p_j]f(G).
\end{aligned}$$

By Lemma 3.1, we apply pebbling moves to all the vertices in $G_{11}, \ldots, G_{1p_1}, G_{21}, \ldots, G_{2p_2}, \ldots, G_{n1}, \ldots, G_{np_n}$ and we can move at least $\sum_{i=1}^{n} \sum_{j=1}^{p_i} (\frac{m_{ij} - r_{ij}}{2})$ pebbles from $G_{11}, \ldots, G_{1p_1}, G_{21}, \ldots, G_{2p_2}, \ldots, G_{n1}, \ldots, G_{np_n}$ to the vertices of $K_n \times G$. Therefore, in $K_n \times G$, we have at least altogether

$$\begin{aligned}
[2(n+1) + \Sigma p_j]f(G) - \sum_{i=1}^{n} \sum_{j=1}^{p_i} m_{ij} + \sum_{i=1}^{n} \sum_{j=1}^{p_i} \left(\frac{m_{ij} - r_{ij}}{2}\right) &= [2(n+1) + \Sigma p_j]f(G) - \sum_{i=1}^{n} \sum_{j=1}^{p_i} \left(\frac{m_{ij} + r_{ij}}{2}\right) \\
&\ge [2(n+1) + \Sigma p_j]f(G) - (n + 1 + \Sigma p_j)f(G) \\
&= (n+1)f(G)
\end{aligned}$$

pebbles. Since $f(K_n \times G, (v_1, y)) \le (n+1)f(G)$, then we can move one pebble to $(v_1, y)$.

Next we assume that the target vertex is $(u_{ij}, y)$ for some $y$, where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, p_i$. Without loss of generality, we assume that the target vertex is $(u_{11}, y)$. If we delete all edges between the vertex $v_i$ ($i = 2, \ldots, n$) and $v_j$ ($j = 2, \ldots, n$) in the graph $K_n^*$, we get a tree $T$. By Lemma 3.2, we know that $f(T, u_{11}) = 2(n+1) + \Sigma p_j$. By Lemma 3.3, we know that $f(T \times G, (u_{11}, y)) \le f(T, u_{11})f(G)$. From ref [1], we know that if $G'$ is a spanning subgraph of $G$, then $f(G) \le f(G')$. Since $T$ is a spanning subgraph of $K_n^*$, then $T \times G$ is a spanning subgraph of $K_n^* \times G$. So $f(K_n^* \times G, (u_{11}, y)) \le f(T \times G, (u_{11}, y))$, and consequently $f(K_n^* \times G, (u_{11}, y)) \le [2(n+1) + \Sigma p_j]f(G)$. One pebble can be moved to $(u_{11}, y)$. A thorn graph of a complete graph satisfies the two-pebbling property. The following corollary is obvious. $\square$

**Corollary 4.2.**

$$f(K_n^* \times K_m^*) \le \left[2(n+1) + \sum_{i=1}^{n} p_i\right]\left[2(m+1) + \sum_{j=1}^{m} p_j\right], \quad n > 1, m > 1.$$

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